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Propagation of Covariance Matrices of Track Parameters in Homogeneous Magnetic Fields in CMS

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Abstract

In this paper, a set of Jacobians used for the propagation of track parameter covariance matrices in homogeneous magnetic fields in CMS is derived. Most of the presented formulas have been in widespread use in the high-energy physics community for many years, but have until now only existed in unpublished notes. Very precise, purely numerical schemes for calculating the same derivatives are also presented and used as a baseline for evaluating the correctness of the analytical terms.

1 Introduction

Transport or propagation of track parameters and of the associated covariance matrices are vital parts of track reconstruction algorithms. This is due to the fact that most reconstruction algorithms are based on a least-squares fit, and in such an estimation method the track parameters and the associated covariance matrix are used. Given an estimate of the parameters and their uncertainties at some point in a track detector, transporting these to a detector unit containing a measurement is a necessary prerequisite for inclusion of the measurement into the fit. In general, such a transport can logically be divided into two different steps. The first deals with the purely geometrical propagation governed by the equations of motion. The second procedure takes material effects such as energy loss and multiple Coulomb scattering into account. In this paper, the first part of this two-fold procedure will be addressed.

In a homogeneous magnetic field the particle trajectory is a helix, implying that the track parameters are analytical functions of the path length. In a track reconstruction context one often deals with propagation under constraints, i.e. propagation between cylindrical or planar surfaces, and one of the tasks for the parameter propagation is to find an appropriate expression of the path length. For propagation between concentric cylinders or between parallel planes exact expressions of the path length are available, whereas for propagation between planes of arbitrary orientation one has to resort to an iterative approach. In this paper it will be assumed that the problem of finding the path length has been adequately solved.

The track parameters at one detector layer are in general non-linear functions of the parameters at another layer, i.e. the parameter propagation is done non-linearly. In this case it is not obvious how to propagate the associated covariance matrices. The common solution to this problem is to expand the parameter propagation functions to first order in a Taylor series and use these derivatives to propagate the covariance matrices in an approximate way. Such a procedure is called *linear error propagation*. The availability of the derivatives of the propagated track parameters with respect to those at the starting point of the propagation – so-called *Jacobians* – is therefore essential for a successful error propagation. As for the parameter propagation, the calculations of these Jacobians are simpler when restricting to propagation between concentric cylinders or parallel planes [1] than between planes of arbitrary orientation.

The main purpose of this paper is to derive and present the relevant Jacobians needed for the latter case. Until now these derivations have only existed in unpublished notes [2, 3]. The formulas have been extensively used in the high-energy physics community during the last 25 years, for instance in the error propagation part `GEANE` [4] of the simulation package `GEANT3`, and in the current reconstruction software of the CMS tracker at the LHC. We also present a direct evaluation of the correctness and precision of the Jacobians by means of a very precise, purely numerical algorithm for the calculation of the derivatives. As a result, two terms suffering from numerical instabilities have been identified. These instabilities particularly occur in the high-momentum range when using single precision floating point numbers[1)] in the propagation. Approximate, numerically stable formulas have been derived and are presented.

The problem of calculating transport Jacobians from one plane of arbitrary orientation to another naturally decomposes into three separate parts. This is because the error propagation from one spatial location to another most easily is done in a coordinate frame which moves along with the track – a so-called *curvilinear* frame. The natural decomposition is therefore first a transformation from a local coordinate system at the starting surface to the curvilinear frame, then a transport within the curvilinear frame to the destination surface, and finally a transformation from the curvilinear frame to a local frame at the destination surface. The total Jacobian becomes the matrix product of the three different Jacobians. Expressions of Jacobians addressing the same problem exist in the literature [5], valid for small-step propagations and for high-momentum particles. Our calculations are valid for propagations of any step length and include correlations introduced by the track curvature, thereby making the expressions valid for all momenta.

The paper is organized as follows. In Section 2, general expressions of total differentials with respect to position and direction are derived, leading naturally to the above-mentioned transformations as different, special cases. Another special case mentioned in this section is the transformations between the curvilinear and the so-called *perigee* frame [6], which is useful when propagation of errors to the point of closest approach to a reference point is desired. In Section 3, the results of the numerical evaluation of the Jacobians are presented. Most of the specific expressions of these Jacobians are compiled in an appendix.

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1) The IEEE standard – implemented on many computers – defines a single precision floating point number to be 32 bits with a relative precision of about 10^{-7}. A 64-bit double precision floating point number has a relative precision of about 10^{-16}.

2 Error propagation

The natural starting point for a general description of propagation of errors along a helix is the equation of the trajectory of a charged particle in a homogeneous magnetic field [2, 3],

$$\mathbf{M} = \mathbf{M}_0 + \frac{\gamma}{Q}(\theta - \sin\theta)\cdot\mathbf{H} + \frac{\sin\theta}{Q}\cdot\mathbf{T}_0 + \frac{\alpha}{Q}(1-\cos\theta)\cdot\mathbf{N}_0, \tag{1}$$

with $\mathbf{M}$ being the position vector of the point on the helix at path length s from the reference point $\mathbf{M}_0$ (at $s=0$), $\mathbf{H} = \mathbf{B}/|\mathbf{B}|$ being a normalized magnetic field vector, $\mathbf{T} = \mathbf{p}/|\mathbf{p}|$ being a normalized tangent vector to the track, $\mathbf{N} = (\mathbf{H}\times\mathbf{T})/\alpha$ with $\alpha = |\mathbf{H}\times\mathbf{T}|$, $\gamma = \mathbf{H}\cdot\mathbf{T}$, $Q = -|\mathbf{B}|\,q/p$ with $p = |\mathbf{p}|$ being the absolute value of the 3-momentum vector, $q = \pm 1$ denoting the charge of the particle, and $\theta = Q\cdot s$. The numerical value of $|\mathbf{B}|$ is $0.3\cdot 10^{-3}$ times the field in units of kGauss if p is given in units of GeV and the path length s is given in units of cm. In the following, a subscript 0 indicates quantities defined at the starting point $s=0$. Any point along the trajectory can be specified by a corresponding value of s.

The equation of the tangent vector $\mathbf{T}$ is found by differentiating Eq. (1) with respect to s,

$$\mathbf{T} = \frac{\partial\mathbf{M}}{\partial s} = \gamma(1-\cos\theta)\cdot\mathbf{H} + \cos\theta\cdot\mathbf{T}_0 + \alpha\sin\theta\cdot\mathbf{N}_0. \tag{2}$$

Next, the differentials $d\mathbf{M}$ and $d\mathbf{T}$ will be formed. These differentials will constitute the key equations which relate variations of position, direction and momentum at the starting point ($s=0$) to variations of the same quantities at any other point along the helix. The differentials are given by

$$\begin{aligned} d\mathbf{M} &= \frac{\partial\mathbf{M}}{\partial\mathbf{M}_0}\cdot d\mathbf{M}_0 + \frac{\partial\mathbf{M}}{\partial\mathbf{T}_0}\cdot d\mathbf{T}_0 + \frac{\partial\mathbf{M}}{\partial(q/p_0)}\cdot\delta(q/p_0) + \frac{\partial\mathbf{M}}{\partial s}\cdot\delta s, & (3)\\ d\mathbf{T} &= \frac{\partial\mathbf{T}}{\partial\mathbf{T}_0}\cdot d\mathbf{T}_0 + \frac{\partial\mathbf{T}}{\partial(q/p_0)}\cdot\delta(q/p_0) + \frac{\partial\mathbf{T}}{\partial s}\cdot\delta s, & (4)\end{aligned}$$

where $\delta(q/p_0)$ is the variation of the charged, inverse momentum at the starting point and δs is the change in path length of the helix due to the variations at the starting point. An illustration of this effect is shown in Fig. 1. The

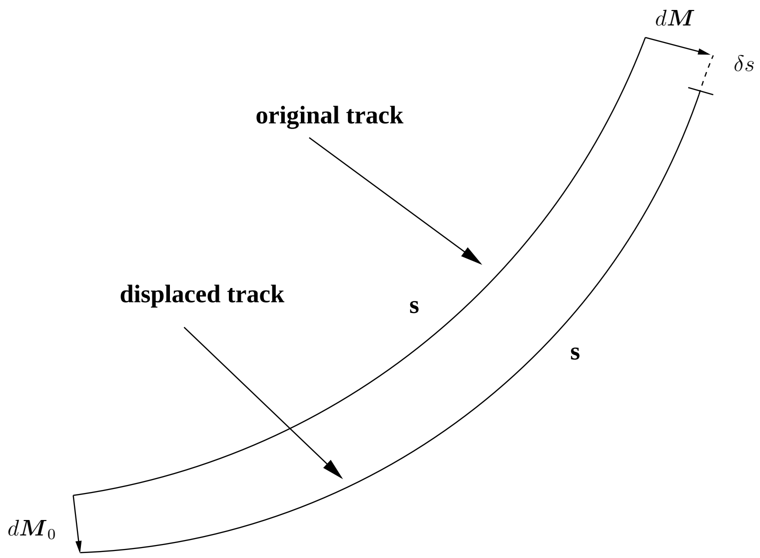

Figure 1: A track and the displaced track due to a variation $d\mathbf{M}_0$ are shown. In the error propagation the change δs of the path length has to be taken into account. In this specific case $d\mathbf{M}$ is understood to be perpendicular to the track (see also Eq. (21)).

partial derivatives are obtained by direct differentiation of Eq. (1) and (2), and the results are

$$\frac{\partial \mathbf{M}}{\partial \mathbf{M}_0} \cdot d\mathbf{M}_0 = d\mathbf{M}_0, \tag{5}$$

$$\begin{aligned} \frac{\partial \mathbf{M}}{\partial \mathbf{T}_0} \cdot d\mathbf{T}_0 &= \frac{\theta - \sin\theta}{Q} \cdot (\mathbf{H} \cdot d\mathbf{T}_0) \cdot \mathbf{H} + \frac{\sin\theta}{Q} \cdot d\mathbf{T}_0 \\ &\quad + \frac{1-\cos\theta}{Q} \cdot (\mathbf{H} \times d\mathbf{T}_0), \end{aligned} \tag{6}$$

$$\frac{\partial \mathbf{M}}{\partial (q/p_0)} \cdot \delta (q/p_0) = \frac{1}{q/p} [s \cdot \mathbf{T} + \mathbf{M}_0 - \mathbf{M}] \cdot \delta (q/p_0), \tag{7}$$

$$\frac{\partial \mathbf{M}}{\partial s} \cdot \delta s = \mathbf{T} \cdot \delta s, \tag{8}$$

$$\begin{aligned} \frac{\partial \mathbf{T}}{\partial \mathbf{T}_0} \cdot d\mathbf{T}_0 &= \cos\theta \cdot d\mathbf{T}_0 + (1-\cos\theta) \cdot (\mathbf{H} \cdot d\mathbf{T}_0) \cdot \mathbf{H} \\ &\quad + \sin\theta \cdot (\mathbf{H} \times d\mathbf{T}_0), \end{aligned} \tag{9}$$

$$\frac{\partial \mathbf{T}}{\partial (q/p_0)} \cdot \delta (q/p_0) = \frac{\alpha Q s}{q/p} \cdot \mathbf{N} \cdot \delta (q/p_0), \tag{10}$$

$$\frac{\partial \mathbf{T}}{\partial s} \cdot \delta s = \alpha Q \cdot \mathbf{N} \cdot \delta s. \tag{11}$$

2.1 Transformation from one curvilinear frame to another

The curvilinear frame is uniquely defined at each point along the track by three orthogonal unit vectors $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{T}$, defining a coordinate system $(x_\perp, y_\perp, z_\perp)$. The vector $\mathbf{T}$ has been defined above as the unit vector parallel to the track, pointing in the particle direction. The two vectors $\mathbf{U}$ and $\mathbf{V}$ are defined by

$$\mathbf{U} = \frac{\mathbf{Z} \times \mathbf{T}}{|\mathbf{Z} \times \mathbf{T}|}, \tag{12}$$

$$\mathbf{V} = \mathbf{T} \times \mathbf{U}, \tag{13}$$

where $\mathbf{Z}$ is the unit vector pointing in the direction of the global z-axis. This means that the $z_\perp$-axis is pointing along the particle direction, the $x_\perp$-axis is lying in the global xy-plane, while the $y_\perp$-axis is given by the requirement that the three axes should form a Cartesian, right-handed coordinate system. In addition, the curvilinear set of parameters includes q/p, the dip angle λ of the particle 3-momentum vector and the angle of inclination ϕ between the tangent of the projection of the particle 3-momentum vector into the global xy-plane and the global x-axis. Even though the angles are defined at the point of intersection between the particle 3-momentum vector and the curvilinear plane, their values are given in the global, Cartesian reference frame (X, Y, Z). The relations between the momentum components (p_x, p_y, p_z) in the Cartesian frame and the angles are

$$p_x = p\cos\lambda\cos\phi, \tag{14}$$

$$p_y = p\cos\lambda\sin\phi, \tag{15}$$

$$p_z = p\sin\lambda. \tag{16}$$

The Jacobian of the transformation from a curvilinear frame $(q/p, \lambda, \phi, x_\perp, y_\perp)$ at $s_0 = 0$ to the same set of parameters at path length s is then derived by forming the differentials $d\mathbf{M}$ and $d\mathbf{T}$, introducing the specific constraints given by the curvilinear frames,

$$d\mathbf{M}_0 = \mathbf{U}_0 \cdot \delta x_{\perp 0} + \mathbf{V}_0 \cdot \delta y_{\perp 0}, \tag{17}$$

$$d\mathbf{T}_0 = \frac{\partial \mathbf{T}_0}{\partial \lambda_0} \cdot \delta\lambda_0 + \frac{\partial \mathbf{T}_0}{\partial \phi_0} \cdot \delta\phi_0 = \mathbf{V}_0 \cdot \delta\lambda_0 + \cos\lambda_0 \cdot \mathbf{U}_0 \cdot \delta\phi_0, \tag{18}$$

$$d\mathbf{M} = \mathbf{U} \cdot \delta x_\perp + \mathbf{V} \cdot \delta y_\perp, \tag{19}$$

$$d\mathbf{T} = \mathbf{V} \cdot \delta\lambda + \cos\lambda \cdot \mathbf{U} \cdot \delta\phi. \tag{20}$$

Also, since $d\mathbf{M}$ now is defined to be a variation in a plane perpendicular to the track, the functional dependence of δs on the variations of position, direction and momentum at the starting point can be evaluated by multiplying Eq. (3) with $\mathbf{T}$ and using the constraint $d\mathbf{M} \cdot \mathbf{T} = 0$. One obtains

$$\delta s = -\mathbf{T} \cdot d\mathbf{M}_0 - \mathbf{T} \cdot \left(\frac{\partial \mathbf{M}}{\partial \mathbf{T}_0} \cdot d\mathbf{T}_0 \right) - \left(\mathbf{T} \cdot \frac{\partial \mathbf{M}}{\partial (q/p_0)} \right) \cdot \delta (q/p_0). \tag{21}$$

By inserting Eq. (21) and Eq. (5)-(11) into Eq. (3) and (4), making use of Eq. (17)-(20), we get a set of equations relating variations of the parameters at the starting surface to the variations at the destination surface. These can then quite straightforwardly be manipulated to yield the differentials of the parameters at the destination surface. Since, for instance, the differential $\delta x_\perp$ is defined as

$$\delta x_\perp = \frac{\partial x_\perp}{\partial (q/p_0)} \cdot \delta (q/p_0) + \frac{\partial x_\perp}{\partial \lambda_0} \cdot \delta\lambda_0 + \frac{\partial x_\perp}{\partial \phi_0} \cdot \delta\phi_0 + \frac{\partial x_\perp}{\partial x_{\perp 0}} \cdot \delta x_{\perp 0} + \frac{\partial x_\perp}{\partial y_{\perp 0}} \cdot \delta y_{\perp 0}, \tag{22}$$

the different terms in the desired Jacobian can be identified as the multiplying factors of the variations at the starting surface. A complete list of these can be found in the appendix. Since we assume a perfectly helical track model, the momentum at the destination surface is equal to the momentum at the starting surface.

2.2 Transformations between curvilinear and local frames at a fixed point on the particle trajectory

Let us first consider the transformation from a local plane – defined by a unit vector $\mathbf{I}$ normal to the plane and two, orthogonal unit vectors $\mathbf{J}$ and $\mathbf{K}$ inside the plane – to the curvilinear frame. These three unit vectors define a coordinate system (u, v, w), and a set of parameters describing a track in such a local frame is $(q/p, v', w', v, w)$, where $v' = dv/du$ and $w' = dw/du$, respectively. The desire is to derive the Jacobian of the transformation between $(q/p, v', w', v, w)$ and $(q/p, \lambda, \phi, x_\perp, y_\perp)$ at a given point s on the particle trajectory. The relevant differentials are now

$$\begin{aligned} d\mathbf{M} &= \mathbf{U} \cdot \delta x_\perp + \mathbf{V} \cdot \delta y_\perp = \mathbf{J} \cdot \delta v + \mathbf{K} \cdot \delta w + \mathbf{T} \cdot \delta s, & (23) \\ d\mathbf{T} &= \mathbf{V} \cdot \delta\lambda + \cos\lambda \cdot \mathbf{U} \cdot \delta\phi = \frac{\partial \mathbf{T}}{\partial v'} \cdot \delta v' + \frac{\partial \mathbf{T}}{\partial w'} \cdot \delta w' + \frac{\partial \mathbf{T}}{\partial s} \cdot \delta s. & (24) \end{aligned}$$

Since again $d\mathbf{M}$ is orthogonal to $\mathbf{T}$, δs can be calculated from Eq. (23), in a similar manner as before. The result is

$$\delta s = -(\mathbf{T} \cdot \mathbf{J}) \cdot \delta v - (\mathbf{T} \cdot \mathbf{K}) \cdot \delta w. \tag{25}$$

An example of a change in path length δs induced by a change in one of the local coordinates is shown in Fig. 2.

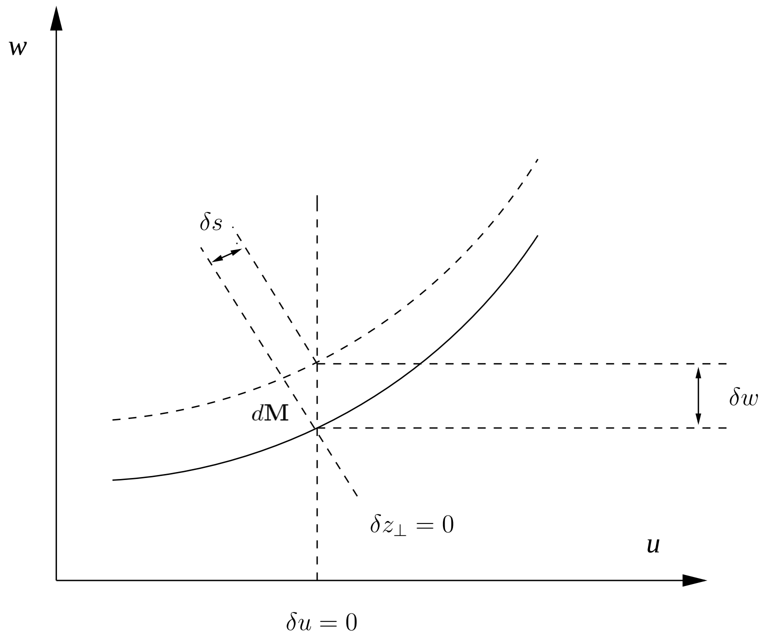

Figure 2: A track and the displaced track due to a variation δw are shown. In order to fulfil the condition $\delta z_\perp = 0$, the track has to undergo a change δs in path length.

By inserting Eq. (25) into Eq. (23) and (24) and following the same procedure as in Section 2.1, explicit expressions of the differentials can again be constructed. For this we also need $\mathbf{T}$ and its derivatives expressed in the local parameters. The formulas are

$$\mathbf{T} = \frac{1}{\sqrt{1+v'^2+w'^2}}\left[\mathbf{I}+v'\mathbf{J}+w'\mathbf{K}\right], \tag{26}$$

$$\frac{\partial \mathbf{T}}{\partial v'} = \frac{1}{\sqrt{1+v'^2+w'^2}}\left[\mathbf{J}-\frac{v'}{\sqrt{1+v'^2+w'^2}}\mathbf{T}\right], \tag{27}$$

$$\frac{\partial \mathbf{T}}{\partial w'} = \frac{1}{\sqrt{1+v'^2+w'^2}}\left[\mathbf{K}-\frac{w'}{\sqrt{1+v'^2+w'^2}}\mathbf{T}\right]. \tag{28}$$

The Jacobian of the transformation from the curvilinear to the local frame can be derived by inverting the Jacobian of the transformation from the local to the curvilinear frame. The expressions of both Jacobians can be found in the appendix.

2.3 Transformations between curvilinear and perigee frames

The so-called perigee frame [6] – useful for vertex reconstruction – is defined by the set of parameters $(\kappa, \theta, \phi, \epsilon, z_p)$, where $\kappa = -qB_z/p_T$ is the (signed) curvature with B_z being the z-component of the magnetic field, $p_T = p\sin\theta$ is the transverse momentum, $\theta = \pi/2 - \lambda$ is the polar angle, ϕ is the angle of inclination in the transverse plane, while ϵ and z_p are coordinates in a local coordinate system to be defined below. This set of parameters is defined at the point P of closest approach in the transverse plane to a reference point, usually close to an expected vertex position. The reference point becomes the origin of a global, Cartesian coordinate system (X, Y, Z). An illustration can be found in Fig. 3. The direction of the magnetic field is usually chosen to coincide with the direction of the global z-axis, but there might be small deviations due to e. g. misalignment effects. The term “transverse” always refers to quantities defined in the (X, Y)-plane, even if the magnetic field direction does not exactly coincide with the direction of the global z-axis.

The perigee to curvilinear transformation can be regarded as a special case of a local to curvilinear transformation for what considers the position coordinates. The parameters ϵ and z_p are coordinates in a local, Cartesian frame defined by the set of unit vectors $(\mathbf{I}, \mathbf{J}, \mathbf{K})$ and with origin equal to the global origin:

$$\mathbf{J} = \frac{\mathbf{T}\times\mathbf{Z}}{|\mathbf{T}\times\mathbf{Z}|} = -\mathbf{U}, \tag{29}$$

$$\mathbf{K} = \mathbf{Z}, \tag{30}$$

$$\mathbf{I} = \mathbf{J}\times\mathbf{K}, \tag{31}$$

where $\mathbf{J}$ and $\mathbf{K}$ are the unit vectors defining the ϵ- and z_p-axes, respectively. $\mathbf{J}$ is in the transverse plane, orthogonal to the projection into the global xy-plane of the tangent vector to the track. $\mathbf{K}$ points along the global z-axis. In order to yield a right-handed frame, $\mathbf{I}$ must be anti-parallel to the projected tangent vector. Thus, it follows that the sign of ϵ is the same as the sign of the z-component of the angular momentum of the particle with respect to the reference point, and z_p is equal to the z-component of P in the global frame. The definition of κ is such that the sign is positive for a counterclockwise rotation of the particle (which is the sense of rotation of a negatively charged particle). These definitions – including the sign conventions – are identical to those described in the paper by Billoir and Qian [6], and we refer the interested reader to this for more detailed explanations.

With these definitions kept in mind, differentials can be formed, giving

$$d\mathbf{M} = \mathbf{U}\cdot\delta x_\perp + \mathbf{V}\cdot\delta y_\perp = \mathbf{J}\cdot\delta\epsilon + \mathbf{K}\cdot\delta z_p + \mathbf{T}\cdot\delta s, \tag{32}$$

$$d\mathbf{T} = \mathbf{V}\cdot\delta\lambda + \cos\lambda\cdot\mathbf{U}\cdot\delta\phi = -\mathbf{V}\cdot\delta\theta + \cos\lambda\cdot\mathbf{U}\cdot\delta\phi + \frac{\partial\mathbf{T}}{\partial s}\cdot\delta s, \tag{33}$$

with $\mathbf{T}$, $\mathbf{U}$ and $\mathbf{V}$ having the same meaning as before. It should be emphasized that the variations $(\delta\lambda, \delta\phi)$ on the left-hand side of (33) are variations at fixed $z_\perp$, whereas the variations $(\delta\theta, \delta\phi)$ on the right-hand side are variations at fixed u. In general they are different, except in the case $\delta\epsilon = \delta z_p = 0$.

The expression of δs becomes

$$\delta s = -(\mathbf{T}\cdot\mathbf{J})\cdot\delta\epsilon - (\mathbf{T}\cdot\mathbf{K})\cdot\delta z_p. \tag{34}$$

Following exactly the same procedure as before, all terms of the Jacobian relating positions and directions can straightforwardly be derived. The only terms missing are now those related to momentum and curvature, and they

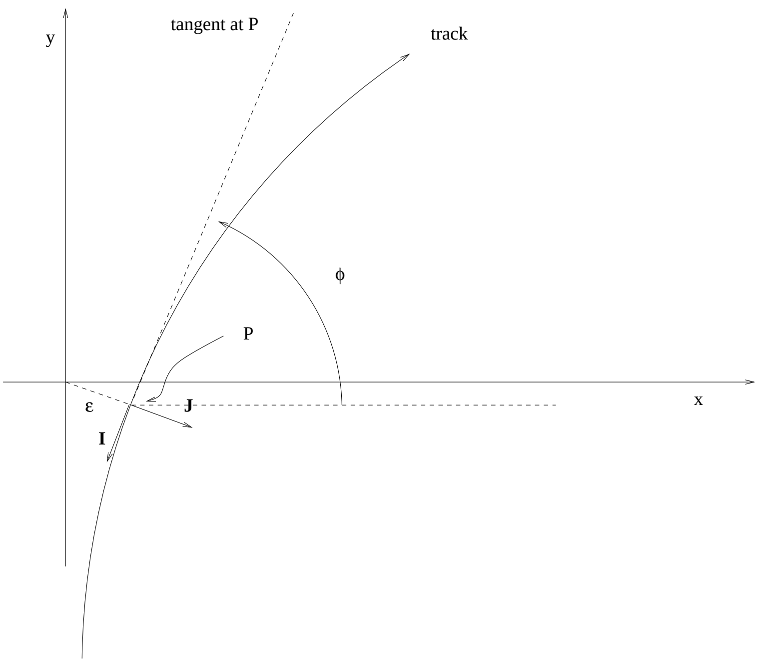

Figure 3: Definition of the parameters ϵ and ϕ, as well as the directions of the unit vectors $\mathbf{I}$ and $\mathbf{J}$. The origin of the coordinate system defined by the set of unit vectors $(\mathbf{I}, \mathbf{J}, \mathbf{K})$ is equal to the global origin. The point P of closest approach in the transverse plane is also shown.

can be identified by the relation

$$\frac{q}{p} = -\frac{\sin\theta}{B_z}\kappa, \tag{35}$$

yielding

$$\delta\left(\frac{q}{p}\right) = -\frac{\sin\theta}{B_z} \cdot \delta\kappa - \frac{\cos\theta}{B_z}\kappa \cdot \delta\theta. \tag{36}$$

The Jacobian of the transformation from the curvilinear to the perigee frame can be derived by inverting the Jacobian of the transformation from the perigee to the curvilinear frame. The Jacobians of both transformations can again be found in the appendix.

It can be noted that the curvilinear frame under consideration in this section is defined at the point P. If the covariance matrix is desired somewhere else along the track, it has to be further propagated within the curvilinear frame itself, as described in Section 2.1.

2.4 Transformations between global Cartesian and local frames

The global Cartesian frame (p_x, p_y, p_z, x, y, z) is useful for vertex reconstruction purposes as well as for track reconstruction in zero magnetic fields. The latter situation is typical in for instance test-beam applications.

When invoking the global, Cartesian frame in the CMS track reconstruction code [7], the basic strategy is to go via a Cartesian frame which is aligned with the local or curvilinear frame under consideration. This Cartesian frame is related to the global Cartesian frame via a pure rotation. Any attempt to use the global Cartesian system as an intermediate frame for transformations between different five-dimensional systems can therefore be understood to amount to a sequence of rotations, and in this procedure the notion of the path length changing during the transformation is lost. Such an approach will thus only be correct in the special case of transforming between two parallel planes, since in this case the path length does not change. In other situations the correct result – as implemented in the direct transformations between local and curvilinear frames – is not obtained.

First the transformations between a local frame and the global, Cartesian frame are considered. The intermediate, Cartesian frame will be denoted by primed quantities (p'_x, p'_y, p'_z, x', y', z'), where the x'- and y'-axes are parallel to the v- and w-axes of the local frame, respectively. The Jacobian $\mathbf{R}$ of the transformation from the global, Cartesian frame to the primed system consists of two similar, three-by-three blocks, each consisting of the relevant rotation matrix. The other entries of this matrix are zero. The Jacobian $\mathbf{J}_{C'\to l}$ of the transformation from the primed, Cartesian system to the local frame is constructed from the following formulas:

$$\frac{q}{p} = \frac{q}{\sqrt{p_x'^2 + p_y'^2 + p_z'^2}}, \tag{37}$$

$$v' = \frac{p_x'}{p_z'}, \tag{38}$$

$$w' = \frac{p_y'}{p_z'}. \tag{39}$$

The total Jacobian is given by the product $\mathbf{J}_{C'\to l} \cdot \mathbf{R}$.

For the inverse transformation, the Jacobian $\mathbf{J}_{l\to C'}$ can be derived from the following relations:

$$p_x' = \frac{q}{q/p} \cdot \frac{\text{sign}\,(p_z) \cdot v'}{\sqrt{1 + v'^2 + w'^2}}, \tag{40}$$

$$p_y' = \frac{q}{q/p} \cdot \frac{\text{sign}\,(p_z) \cdot w'}{\sqrt{1 + v'^2 + w'^2}}, \tag{41}$$

$$p_z' = \frac{q}{q/p} \cdot \frac{\text{sign}\,(p_z)}{\sqrt{1 + v'^2 + w'^2}}, \tag{42}$$

where sign (p_z) is the sign of the z-component of the momentum vector in the local frame. This sign is needed in order to uniquely specify the state of the track in the local frame and is therefore available in the class which contains the information about the local trajectory parameters. The total Jacobian is in this case given by the product $\mathbf{R}^T \cdot \mathbf{J}_{l\to C'}$, using the same matrix $\mathbf{R}$ as above. The Jacobians $\mathbf{J}_{C'\to l}$ and $\mathbf{J}_{l\to C'}$ are shown in the appendix.

2.5 Transformations between global Cartesian and curvilinear frames

The transformation from the curvilinear to the global Cartesian frame is done in two parts. First the curvilinear frame is transformed into a hybrid, Cartesian frame (p_x, p_y, p_z, x', y', z'), where the position axes are parallel to the corresponding axes in the curvilinear frame. The axes of the momentum components, however, are identical to those of the global, Cartesian frame. The following relations,

$$p_x = \frac{q}{q/p} \cos\lambda \cos\phi, \tag{43}$$

$$p_y = \frac{q}{q/p} \cos\lambda \sin\phi, \tag{44}$$

$$p_z = \frac{q}{q/p} \sin\lambda, \tag{45}$$

enable derivations of the Jacobian terms from the curvilinear parameters (q/p, λ, ϕ) to the Cartesian parameters (p_x, p_y, p_z). The transformation from the hybrid, Cartesian frame to the global Cartesian frame is done by multiplying the position part with the relevant rotation matrix and leaving the momentum part untouched.

The inverse transformation is carried out by first transforming from the global Cartesian frame to the hybrid, Cartesian frame in a way similar to the one described above for the other direction. The final transformation from the hybrid, Cartesian frame to the curvilinear frame is done by using the following relations,

$$\frac{q}{p} = \frac{q}{\sqrt{p_x^2 + p_y^2 + p_z^2}}, \tag{46}$$

$$\lambda = \arctan\left(\frac{p_z}{\sqrt{p_x^2 + p_y^2}}\right), \tag{47}$$

$$\phi = \arctan\left(\frac{p_y}{p_x}\right). \tag{48}$$

The Jacobians of the transformations between the hybrid, Cartesian frame and the curvilinear frame can be found in the appendix.

3 Evaluation of error propagation

An evaluation of the most relevant Jacobians for track reconstruction purposes – i.e. those involving transformation within the curvilinear frame as well as transformations between curvilinear and local frames – has been done by using as a baseline a very precise, purely numerical algorithm for the calculations of the derivatives. This algorithm – the so-called Ridders algorithm [8] – considers a standard, symmetric numerical derivative

$$g(h) = \frac{f(x+h) - f(x-h)}{2h} \tag{49}$$

at a certain value of x and regards it as a function of h only. The basic idea is to start out at a reasonably large value of h and calculate the function g at this value plus at a somewhat smaller value of h. If plotted, a line can be drawn through the two points $(h_0, g(h_0))$ and $(h_1, g(h_1))$. Since we are mainly interested in $g(0)$, the intersect between this line and the ordinate axis constitutes a first estimate of the derivative in the limit $h \to 0$. A new, lower value h_2 can be chosen and a parabola drawn through the three points available. The second estimate of the desired derivative is therefore the value of this parabola at $h = 0$. This procedure can be repeated until a fixed number of step sizes h has been processed or until the difference between the extrapolation to $h = 0$ at a certain degree of the polynomial and at one degree higher is smaller than a given tolerance. This difference constitutes a measure of the error in the calculation of the derivative. The accuracy of the derivative is approximately the same as the accuracy of f itself. Fig. 4 shows a schematic plot of a function with four different values of h and the extrapolation to

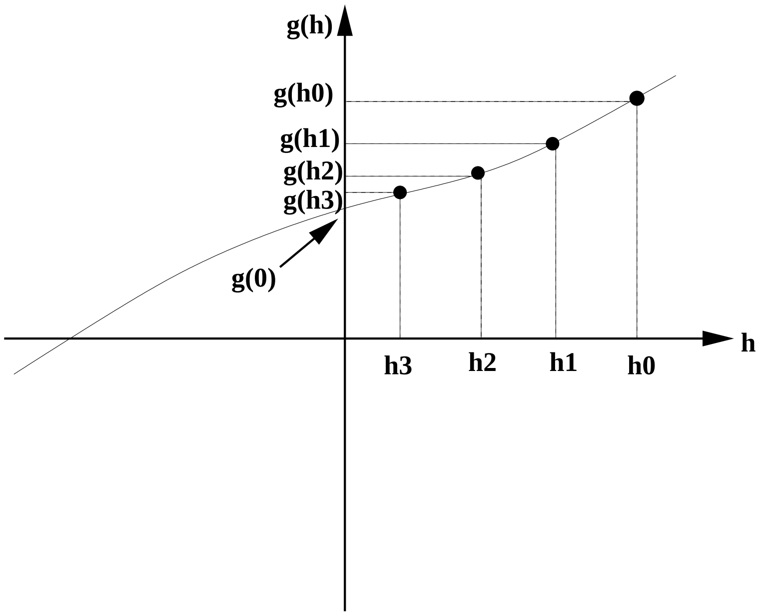

Figure 4: Schematic plot of function $g(h)$.

the value at $h = 0$. For the purpose of calculating Jacobians there are in total 25 different functions f, namely functions relating any of the five different track parameters at the starting point of the propagation to any of the five parameters at the destination point.

The evaluation has been based on a set of 10000 simulated tracks in a setup which resembles the CMS tracker. Charged particles have been generated at random positions close to the origin of a global coordinate system. From these starting points, the particles have been propagated along helix curves random path lengths between 0 and 50 cm in random directions between pseudorapidities $\eta = -2.5$ and $\eta = 2.5$. The propagation of the track parameters has been done by using the standard helix propagator in the CMS reconstruction software [7]. At the starting and destination points of the propagation, planes with random tilts with respect to the plane normal to the particle direction have been created. The numerical derivatives of the track parameters in the local frame at the destination surface with respect to those in the local frame at the starting surface define the baseline of evaluation of the corresponding terms in the analytical Jacobian. The range of the transverse momenta is from 0.9 GeV/c to 1000 GeV/c, generated from a flat distribution. The helix propagator uses by default single precision arithmetics, but for the calculations of the numerical derivatives a special version using double precision has been applied.

The outcome of such a procedure for one specific term in the Jacobian can be found in Fig. 5. Here a histogram of

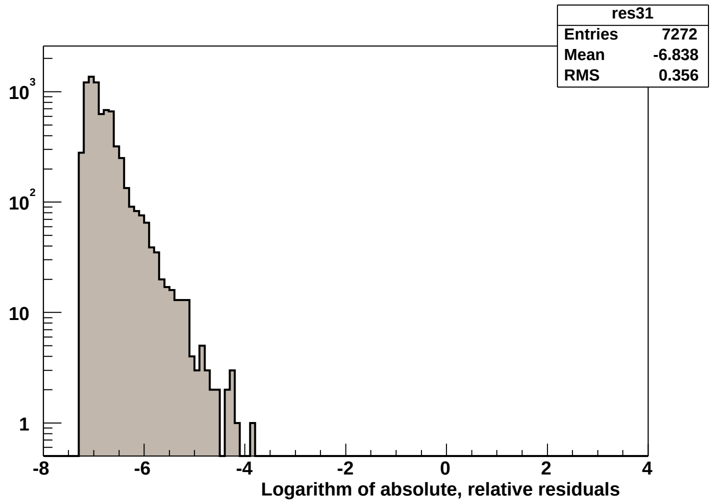

Figure 5: Logarithm of the absolute, relative residual between analytical and numerical derivative for a typical stable term.

the logarithm of the absolute, relative difference between the value from the analytical Jacobian and the numerical result as given by the Ridders algorithm is shown. Due to the underlying single precision of the analysis program used to produce the plot, PAW [9], no relative difference better than approximately 10^{-7} is seen. Entries with better relative precision become identically zero and are not shown. Quantities such as positions and rotation matrices for the various surfaces involved have been calculated in single precision, which is also the default choice in the CMS reconstruction software. The calculations of the baseline numerical derivatives are carried out in double precision also for these quantities. The precision displayed in Fig. 5 is therefore to be expected and is indeed exhibited for the vast majority of the Jacobian terms. However, two of the terms – the derivatives of the local positions at the destination surface with respect to the charged inverse momentum at the starting surface – behave differently. A histogram with the relative precision of one of those is shown in Fig. 6. This term is significantly less precise than

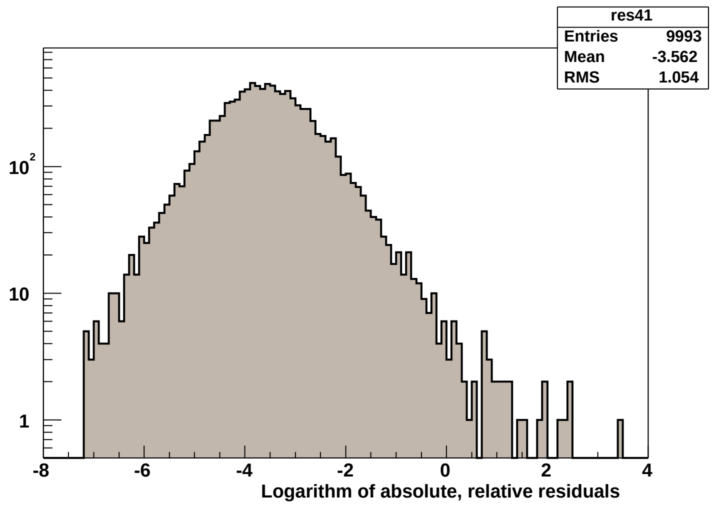

Figure 6: Logarithm of the absolute, relative residual between analytical and numerical derivative for one of the unstable terms.

the one shown in Fig. 5, and the loss of precision comes from the derivatives of the curvilinear local positions at the destination surface with respect to the charged, inverse momentum at the starting surface. In the expressions of these numerically unstable terms there are summations of reasonably large numbers, whose result can be very close to zero. This quantity is again multiplied by a potentially large number to yield the final result. Due to the single precision arithmetics, the instability originates from the procedure which generates the close-to-zero temporary quantity.

Numerically stable terms have been derived by expanding the $\sin\theta$ and $\cos\theta$ terms in Eq. (1) in Taylor series up to third and fourth order in θ, respectively, and using the modified expression for $\mathbf{M}$ as a starting point for the calculations of the derivatives. The calculations have been carried out in exactly the same way as described in Section 2, and the resulting expressions can be found in Appendix 4. It can be noted that equivalent expressions could have been obtained by directly expanding the terms of the relevant derivatives.

The instability occurs for those terms which – at low s – depend quadratically on the path length s. It is mainly visible for propagations over small values of θ – i.e. for small propagation distances or high momenta or both – so for each propagation a test is made on the actual value of θ. For values of θ smaller than a suitable limit, the approximate, numerically stable terms are chosen. For values of θ larger than the limit the standard formulas are applied. A histogram of the resulting relative precision is shown in Fig. 7. As a guide to the eye, the histogram in

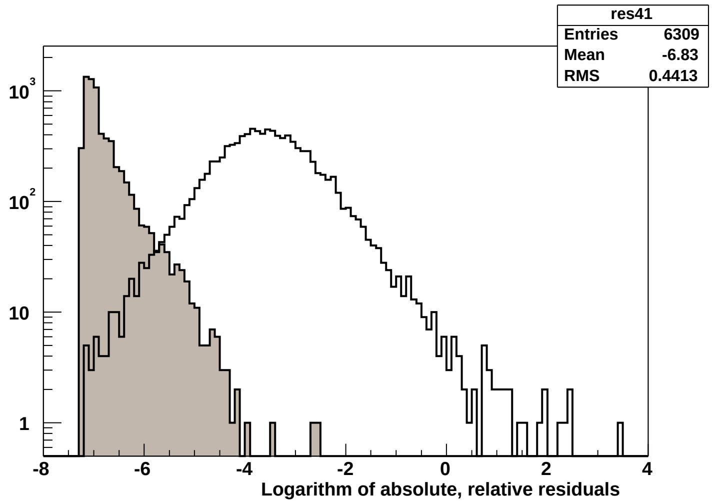

Figure 7: Logarithm of the absolute, relative residual between analytical and numerical derivative for one of the numerically stabilized terms. The histogram for the corresponding unstable term is shown as a guide to the eye.

Fig. 6 is also included.

The same instabilities are also visible when using double precision in the calculation of the analytical derivatives, but to a much lesser extent. In this case only a small percentage of the terms exhibit relative precision worse than the lower limit visible in `PAW`, whereas the corresponding, numerically stable terms are all better than this limit. The limiting value of θ will also be much lower.

4 Conclusions

A set of Jacobians used for propagation of covariance matrices of track parameters in homogeneous magnetic fields in CMS has been derived, allowing for propagation between planes of arbitrary orientation. Independent and very precise, purely numerical calculations of the same derivatives have been presented and used to evaluate the correctness and precision of the analytical terms. As a result of the evaluation, two terms suffering from numerical instabilities when using single precision arithmetics have been identified. Approximate, numerically stable terms have been derived and presented.

Appendix

We first consider the Jacobian of the transformation from one curvilinear frame to another. The non-zero terms of the Jacobian are:

$$\frac{\partial\,(q/p)}{\partial\,(q/p_0)} = 1, \tag{50}$$

$$\frac{\partial\lambda}{\partial\,(q/p_0)} = -\alpha Q\cdot\left(\frac{q}{p}\right)^{-1}\cdot(\mathbf{N}\cdot\mathbf{V})\cdot[\mathbf{T}\cdot(\mathbf{M}_0-\mathbf{M})], \tag{51}$$

$$\begin{aligned}\frac{\partial\lambda}{\partial\lambda_0} ={}& \cos\theta\cdot(\mathbf{V}_0\cdot\mathbf{V})+\sin\theta\cdot((\mathbf{H}\times\mathbf{V}_0)\cdot\mathbf{V})\\ &+(1-\cos\theta)\cdot(\mathbf{H}\cdot\mathbf{V}_0)\cdot(\mathbf{H}\cdot\mathbf{V})\\ &+\alpha\,(\mathbf{N}\cdot\mathbf{V})\,[-\sin\theta\,(\mathbf{V}_0\cdot\mathbf{T})+\alpha\,(1-\cos\theta)\,(\mathbf{V}_0\cdot\mathbf{N})\\ &-(\theta-\sin\theta)\,(\mathbf{H}\cdot\mathbf{T})\,(\mathbf{H}\cdot\mathbf{V}_0)],\end{aligned} \tag{52}$$

$$\begin{aligned}\frac{\partial\lambda}{\partial\phi_0} ={}& \cos\lambda_0\,\{\cos\theta\cdot(\mathbf{U}_0\cdot\mathbf{V})+\sin\theta\cdot((\mathbf{H}\times\mathbf{U}_0)\cdot\mathbf{V})\\ &+(1-\cos\theta)\cdot(\mathbf{H}\cdot\mathbf{U}_0)\cdot(\mathbf{H}\cdot\mathbf{V})\\ &+\alpha\,(\mathbf{N}\cdot\mathbf{V})\,[-\sin\theta\,(\mathbf{U}_0\cdot\mathbf{T})+\alpha\,(1-\cos\theta)\,(\mathbf{U}_0\cdot\mathbf{N})\\ &-(\theta-\sin\theta)\,(\mathbf{H}\cdot\mathbf{T})\,(\mathbf{H}\cdot\mathbf{U}_0)]\},\end{aligned} \tag{53}$$

$$\frac{\partial\lambda}{\partial x_{\perp 0}} = -\alpha Q\,(\mathbf{N}\cdot\mathbf{V})\,(\mathbf{U}_0\cdot\mathbf{T}), \tag{54}$$

$$\frac{\partial\lambda}{\partial y_{\perp 0}} = -\alpha Q\,(\mathbf{N}\cdot\mathbf{V})\,(\mathbf{V}_0\cdot\mathbf{T}), \tag{55}$$

$$\frac{\partial\phi}{\partial\,(q/p_0)} = -\frac{\alpha Q}{\cos\lambda}\cdot\left(\frac{q}{p}\right)^{-1}\cdot(\mathbf{N}\cdot\mathbf{U})\cdot[\mathbf{T}\cdot(\mathbf{M}_0-\mathbf{M})], \tag{56}$$

$$\begin{aligned}\frac{\partial\phi}{\partial\lambda_0} ={}& \frac{1}{\cos\lambda}\,\{\cos\theta\cdot(\mathbf{V}_0\cdot\mathbf{U})+\sin\theta\cdot((\mathbf{H}\times\mathbf{V}_0)\cdot\mathbf{U})\\ &+(1-\cos\theta)\cdot(\mathbf{H}\cdot\mathbf{V}_0)\cdot(\mathbf{H}\cdot\mathbf{U})\\ &+\alpha\,(\mathbf{N}\cdot\mathbf{U})\,[-\sin\theta\,(\mathbf{V}_0\cdot\mathbf{T})+\alpha\,(1-\cos\theta)\,(\mathbf{V}_0\cdot\mathbf{N})\\ &-(\theta-\sin\theta)\,(\mathbf{H}\cdot\mathbf{T})\,(\mathbf{H}\cdot\mathbf{V}_0)]\},\end{aligned} \tag{57}$$

$$\begin{aligned}\frac{\partial\phi}{\partial\phi_0} ={}& \frac{\cos\lambda_0}{\cos\lambda}\,\{\cos\theta\cdot(\mathbf{U}_0\cdot\mathbf{U})+\sin\theta\cdot((\mathbf{H}\times\mathbf{U}_0)\cdot\mathbf{U})\\ &+(1-\cos\theta)\cdot(\mathbf{H}\cdot\mathbf{U}_0)\cdot(\mathbf{H}\cdot\mathbf{U})\\ &+\alpha\,(\mathbf{N}\cdot\mathbf{U})\,[-\sin\theta\,(\mathbf{U}_0\cdot\mathbf{T})+\alpha\,(1-\cos\theta)\,(\mathbf{U}_0\cdot\mathbf{N})\\ &-(\theta-\sin\theta)\,(\mathbf{H}\cdot\mathbf{T})\,(\mathbf{H}\cdot\mathbf{U}_0)]\},\end{aligned} \tag{58}$$

$$\frac{\partial\phi}{\partial x_{\perp 0}} = -\frac{\alpha Q}{\cos\lambda}\,(\mathbf{N}\cdot\mathbf{U})\,(\mathbf{U}_0\cdot\mathbf{T}), \tag{59}$$

$$\frac{\partial\phi}{\partial y_{\perp 0}} = -\frac{\alpha Q}{\cos\lambda}\,(\mathbf{N}\cdot\mathbf{U})\,(\mathbf{V}_0\cdot\mathbf{T}), \tag{60}$$

$$\frac{\partial x_\perp}{\partial\,(q/p_0)} = \left(\frac{q}{p}\right)^{-1}[\mathbf{U}\cdot(\mathbf{M}_0-\mathbf{M})], \tag{61}$$

$$\begin{aligned}\frac{\partial x_\perp}{\partial\lambda_0} ={}& \frac{\sin\theta}{Q}\,(\mathbf{V}_0\cdot\mathbf{U})+\frac{1-\cos\theta}{Q}\,((\mathbf{H}\times\mathbf{V}_0)\cdot\mathbf{U})\\ &+\frac{\theta-\sin\theta}{Q}\,(\mathbf{H}\cdot\mathbf{V}_0)\cdot(\mathbf{H}\cdot\mathbf{U}),\end{aligned} \tag{62}$$

$$
\begin{aligned}
\frac{\partial x_\perp}{\partial \phi_0} &= \cos\lambda_0 \left\{ \frac{\sin\theta}{Q} (\mathbf{U}_0 \cdot \mathbf{U}) + \frac{1-\cos\theta}{Q} ((\mathbf{H}\times\mathbf{U}_0)\cdot\mathbf{U}) \right. \\
&\quad \left. + \frac{\theta - \sin\theta}{Q} (\mathbf{H}\cdot\mathbf{U}_0)\cdot(\mathbf{H}\cdot\mathbf{U}) \right\}, \qquad (63) \\
\frac{\partial x_\perp}{\partial x_{\perp 0}} &= \mathbf{U}_0 \cdot \mathbf{U}, \qquad (64) \\
\frac{\partial x_\perp}{\partial y_{\perp 0}} &= \mathbf{V}_0 \cdot \mathbf{U}, \qquad (65) \\
\frac{\partial y_\perp}{\partial (q/p_0)} &= \left(\frac{q}{p}\right)^{-1} \left[\mathbf{V}\cdot(\mathbf{M}_0 - \mathbf{M})\right], \qquad (66) \\
\frac{\partial y_\perp}{\partial \lambda_0} &= \frac{\sin\theta}{Q} (\mathbf{V}_0 \cdot \mathbf{V}) + \frac{1-\cos\theta}{Q} ((\mathbf{H}\times\mathbf{V}_0)\cdot\mathbf{V}) \\
&\quad + \frac{\theta - \sin\theta}{Q} (\mathbf{H}\cdot\mathbf{V}_0)\cdot(\mathbf{H}\cdot\mathbf{V}), \qquad (67) \\
\frac{\partial y_\perp}{\partial \phi_0} &= \cos\lambda_0 \left\{ \frac{\sin\theta}{Q} (\mathbf{U}_0 \cdot \mathbf{V}) + \frac{1-\cos\theta}{Q} ((\mathbf{H}\times\mathbf{U}_0)\cdot\mathbf{V}) \right. \\
&\quad \left. + \frac{\theta - \sin\theta}{Q} (\mathbf{H}\cdot\mathbf{U}_0)\cdot(\mathbf{H}\cdot\mathbf{V}) \right\}, \qquad (68) \\
\frac{\partial y_\perp}{\partial x_{\perp 0}} &= \mathbf{U}_0 \cdot \mathbf{V}, \qquad (69) \\
\frac{\partial y_\perp}{\partial y_{\perp 0}} &= \mathbf{V}_0 \cdot \mathbf{V}, \qquad (70)
\end{aligned}
$$

and the two terms numerically stable at small values of θ are given by

$$
\begin{aligned}
\frac{\partial x_\perp}{\partial (q/p_0)} &= -\frac{1}{2} |\mathbf{B}|\, s^2 \cdot (\mathbf{H}\times\mathbf{T}_0)\cdot\mathbf{U} + \frac{1}{3} |\mathbf{B}|^2 s^3 \cdot \frac{q}{p} \cdot (\gamma\mathbf{H} - \mathbf{T}_0)\cdot\mathbf{U} \\
&\quad + \frac{1}{8} |\mathbf{B}|^3 s^4 \cdot \left(\frac{q}{p}\right)^2 \cdot (\mathbf{H}\times\mathbf{T}_0)\cdot\mathbf{U}, \qquad (71) \\
\frac{\partial y_\perp}{\partial (q/p_0)} &= -\frac{1}{2} |\mathbf{B}|\, s^2 \cdot (\mathbf{H}\times\mathbf{T}_0)\cdot\mathbf{V} + \frac{1}{3} |\mathbf{B}|^2 s^3 \cdot \frac{q}{p} \cdot (\gamma\mathbf{H} - \mathbf{T}_0)\cdot\mathbf{V} \\
&\quad + \frac{1}{8} |\mathbf{B}|^3 s^4 \cdot \left(\frac{q}{p}\right)^2 \cdot (\mathbf{H}\times\mathbf{T}_0)\cdot\mathbf{V}. \qquad (72)
\end{aligned}
$$

Then we look at the transformations between curvilinear and local frames. The Jacobian of the transformation from a local frame to the curvilinear frame is given by:

$$
\frac{\partial (q/p, \lambda, \phi, x_\perp, y_\perp)}{\partial (q/p, v', w', v, w)} =
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & (\mathbf{T}\cdot\mathbf{I})(\mathbf{V}\cdot\mathbf{J}) & (\mathbf{T}\cdot\mathbf{I})(\mathbf{V}\cdot\mathbf{K}) & -\alpha Q(\mathbf{T}\cdot\mathbf{J})(\mathbf{V}\cdot\mathbf{N}) & -\alpha Q(\mathbf{T}\cdot\mathbf{K})(\mathbf{V}\cdot\mathbf{N}) \\
0 & \frac{(\mathbf{T}\cdot\mathbf{I})(\mathbf{U}\cdot\mathbf{J})}{\cos\lambda} & \frac{(\mathbf{T}\cdot\mathbf{I})(\mathbf{U}\cdot\mathbf{K})}{\cos\lambda} & -\frac{\alpha Q(\mathbf{T}\cdot\mathbf{J})(\mathbf{U}\cdot\mathbf{N})}{\cos\lambda} & -\frac{\alpha Q(\mathbf{T}\cdot\mathbf{K})(\mathbf{U}\cdot\mathbf{N})}{\cos\lambda} \\
0 & 0 & 0 & (\mathbf{U}\cdot\mathbf{J}) & (\mathbf{U}\cdot\mathbf{K}) \\
0 & 0 & 0 & (\mathbf{V}\cdot\mathbf{J}) & (\mathbf{V}\cdot\mathbf{K})
\end{pmatrix}. \qquad (73)
$$

The inverse Jacobian is given by

$$
\frac{\partial (q/p, v', w', v, w)}{\partial (q/p, \lambda, \phi, x_\perp, y_\perp)} =
$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{(\mathbf{U}\cdot\mathbf{K})}{(\mathbf{T}\cdot\mathbf{I})^2} & \frac{\cos\lambda(\mathbf{V}\cdot\mathbf{K})}{(\mathbf{T}\cdot\mathbf{I})^2} & -\frac{\alpha Q(\mathbf{U}\cdot\mathbf{I})(\mathbf{R}\cdot\mathbf{K})}{(\mathbf{T}\cdot\mathbf{I})^3} & -\frac{\alpha Q(\mathbf{V}\cdot\mathbf{I})(\mathbf{R}\cdot\mathbf{K})}{(\mathbf{T}\cdot\mathbf{I})^3} \\ 0 & \frac{(\mathbf{U}\cdot\mathbf{J})}{(\mathbf{T}\cdot\mathbf{I})^2} & -\frac{\cos\lambda(\mathbf{V}\cdot\mathbf{J})}{(\mathbf{T}\cdot\mathbf{I})^2} & \frac{\alpha Q(\mathbf{U}\cdot\mathbf{I})(\mathbf{R}\cdot\mathbf{J})}{(\mathbf{T}\cdot\mathbf{I})^3} & \frac{\alpha Q(\mathbf{V}\cdot\mathbf{I})(\mathbf{R}\cdot\mathbf{J})}{(\mathbf{T}\cdot\mathbf{I})^3} \\ 0 & 0 & 0 & \frac{(\mathbf{V}\cdot\mathbf{K})}{(\mathbf{T}\cdot\mathbf{I})} & -\frac{(\mathbf{U}\cdot\mathbf{K})}{(\mathbf{T}\cdot\mathbf{I})} \\ 0 & 0 & 0 & -\frac{(\mathbf{V}\cdot\mathbf{J})}{(\mathbf{T}\cdot\mathbf{I})} & \frac{(\mathbf{U}\cdot\mathbf{J})}{(\mathbf{T}\cdot\mathbf{I})} \end{pmatrix}, \tag{74}$$

where $\mathbf{R} = \mathbf{T} \times \mathbf{N}$ and

$$\begin{aligned} (\mathbf{R}\cdot\mathbf{K}) &= (\mathbf{V}\cdot\mathbf{K})(\mathbf{N}\cdot\mathbf{U}) - (\mathbf{U}\cdot\mathbf{K})(\mathbf{N}\cdot\mathbf{V}), & (75) \\ (\mathbf{R}\cdot\mathbf{J}) &= (\mathbf{V}\cdot\mathbf{J})(\mathbf{N}\cdot\mathbf{U}) - (\mathbf{U}\cdot\mathbf{J})(\mathbf{N}\cdot\mathbf{V}). & (76) \end{aligned}$$

The Jacobian of the transformation from the perigee to the curvilinear frame is given by

$$\frac{\partial\left(q/p, \lambda, \phi, x_\perp, y_\perp\right)}{\partial\left(\kappa, \theta, \phi, \epsilon, z_p\right)} =$$

$$\begin{pmatrix} -\frac{\sin\theta}{B_z} & \frac{q}{p\tan\theta} & 0 & 0 & 0 \\ 0 & -1 & 0 & -\alpha Q(\mathbf{T}\cdot\mathbf{J})(\mathbf{V}\cdot\mathbf{N}) & -\alpha Q(\mathbf{T}\cdot\mathbf{K})(\mathbf{V}\cdot\mathbf{N}) \\ 0 & 0 & 1 & -\frac{\alpha Q(\mathbf{T}\cdot\mathbf{J})(\mathbf{U}\cdot\mathbf{N})}{\cos\lambda} & -\frac{\alpha Q(\mathbf{T}\cdot\mathbf{K})(\mathbf{U}\cdot\mathbf{N})}{\cos\lambda} \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & (\mathbf{V}\cdot\mathbf{K}) \end{pmatrix}, \tag{77}$$

while the inverse transformation is given by

$$\frac{\partial\left(\kappa, \theta, \phi, \epsilon, z_p\right)}{\partial\left(q/p, \lambda, \phi, x_\perp, y_\perp\right)} =$$

$$\begin{pmatrix} -\frac{B_z}{\cos\lambda} & -\frac{qB_z\tan\lambda}{p\cos\lambda} & 0 & \frac{qB_z\alpha Q\tan\lambda(\mathbf{U}\cdot\mathbf{I})(\mathbf{N}\cdot\mathbf{V})}{p\cos\lambda(\mathbf{T}\cdot\mathbf{I})} & \frac{qB_z\alpha Q\tan\lambda(\mathbf{V}\cdot\mathbf{I})(\mathbf{N}\cdot\mathbf{V})}{p\cos\lambda(\mathbf{T}\cdot\mathbf{I})} \\ 0 & -1 & 0 & \frac{\alpha Q(\mathbf{U}\cdot\mathbf{I})(\mathbf{N}\cdot\mathbf{V})}{(\mathbf{T}\cdot\mathbf{I})} & \frac{\alpha Q(\mathbf{V}\cdot\mathbf{I})(\mathbf{N}\cdot\mathbf{V})}{(\mathbf{T}\cdot\mathbf{I})} \\ 0 & 0 & 1 & -\frac{\alpha Q(\mathbf{U}\cdot\mathbf{I})(\mathbf{N}\cdot\mathbf{U})}{\cos\lambda(\mathbf{T}\cdot\mathbf{I})} & -\frac{\alpha Q(\mathbf{V}\cdot\mathbf{I})(\mathbf{N}\cdot\mathbf{U})}{\cos\lambda(\mathbf{T}\cdot\mathbf{I})} \\ 0 & 0 & 0 & \frac{(\mathbf{V}\cdot\mathbf{K})}{(\mathbf{T}\cdot\mathbf{I})} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{(\mathbf{T}\cdot\mathbf{I})} \end{pmatrix}. \tag{78}$$

The Jacobian of the transformation from the intermediate, Cartesian frame to the local frame is given by

$$\frac{\partial\left(q/p, v', w', v, w\right)}{\partial\left(p'_x, p'_y, p'_z, x', y', z'\right)} =$$

$$\begin{pmatrix} \frac{-qp'_x}{\left(p'^2_x+p'^2_y+p'^2_z\right)^{3/2}} & \frac{-qp'_y}{\left(p'^2_x+p'^2_y+p'^2_z\right)^{3/2}} & \frac{-qp'_z}{\left(p'^2_x+p'^2_y+p'^2_z\right)^{3/2}} & 0 & 0 & 0 \\ \frac{1}{p'_z} & 0 & -\frac{p'_x}{p'^2_z} & 0 & 0 & 0 \\ 0 & \frac{1}{p'_z} & -\frac{p'_y}{p'^2_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{79}$$

while the Jacobian of the transformation from the local frame to the intermediate, Cartesian frame is given by

$$\frac{\partial\left(p'_x, p'_y, p'_z, x', y', z'\right)}{\partial\left(q/p, v', w', v, w\right)} =$$

$$\begin{pmatrix} -\frac{q\cdot\mathrm{sign}(p_z)}{(q/p)^2}\cdot\frac{v'}{\sqrt{1+v'^2+w'^2}} & \frac{q\cdot\mathrm{sign}(p_z)}{(q/p)}\cdot\frac{1+w'^2}{\left(1+v'^2+w'^2\right)^{3/2}} & -\frac{q\cdot\mathrm{sign}(p_z)}{(q/p)}\cdot\frac{v'w'}{\left(1+v'^2+w'^2\right)^{3/2}} & 0 & 0 \\ -\frac{q\cdot\mathrm{sign}(p_z)}{(q/p)^2}\cdot\frac{w'}{\sqrt{1+v'^2+w'^2}} & -\frac{q\cdot\mathrm{sign}(p_z)}{(q/p)}\cdot\frac{v'w'}{\left(1+v'^2+w'^2\right)^{3/2}} & \frac{q\cdot\mathrm{sign}(p_z)}{(q/p)}\cdot\frac{1+v'^2}{\left(1+v'^2+w'^2\right)^{3/2}} & 0 & 0 \\ -\frac{q\cdot\mathrm{sign}(p_z)}{(q/p)^2}\cdot\frac{1}{\sqrt{1+v'^2+w'^2}} & -\frac{q\cdot\mathrm{sign}(p_z)}{(q/p)}\cdot\frac{v'}{\left(1+v'^2+w'^2\right)^{3/2}} & -\frac{q\cdot\mathrm{sign}(p_z)}{(q/p)}\cdot\frac{w'}{\left(1+v'^2+w'^2\right)^{3/2}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{80}$$

The Jacobian of the transformation from the curvilinear frame to the hybrid, Cartesian frame is given by

$$\frac{\partial\left(p_x, p_y, p_z, x', y', z'\right)}{\partial\left(q/p, \lambda, \phi, x_\perp, y_\perp\right)} =$$

$$\begin{pmatrix} -qp^2\cos\lambda\cos\phi & -p\sin\lambda\cos\phi & -p\cos\lambda\sin\phi & 0 & 0 \\ -qp^2\cos\lambda\sin\phi & -p\sin\lambda\sin\phi & p\cos\lambda\cos\phi & 0 & 0 \\ -qp^2\sin\lambda & p\cos\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{81}$$

while the Jacobian of the transformation from the hybrid, Cartesian frame to the curvilinear frame is given by

$$\frac{\partial\left(q/p, \lambda, \phi, x_\perp, y_\perp\right)}{\partial\left(p_x, p_y, p_z, x', y', z'\right)} =$$

$$\begin{pmatrix} -\frac{qp_x}{p^3} & -\frac{qp_y}{p^3} & -\frac{qp_z}{p^3} & 0 & 0 & 0 \\ -\frac{p_xp_z}{p_Tp^2} & -\frac{p_yp_z}{p_Tp^2} & \frac{p_T}{p^2} & 0 & 0 & 0 \\ -\frac{p_y}{p_T^2} & \frac{p_x}{p_T^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{82}$$

where $p_T = \sqrt{p_x^2 + p_y^2}$.

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