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Error analysis of the track fit on the Riemann sphere

A. Strandlie[a,*], R. Frühwirth[b]

[a] Faculty of Technology, Gjøvik College, P.O. Box 191, N-2802 Gjøvik, Norway

[b] Institute for High Energy Physics of the Austrian Academy of Sciences, Vienna, Austria

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Abstract

We present in this paper a derivation of the covariance matrix of the estimated track parameters given by the Riemann sphere track fitting method. Results of a simulation experiment from the ATLAS Transition Radiation Tracker show that the covariance matrix presented herein very well reflects the actual spread of the track parameters.

Introduction

At the next generation of high-energy physics experiments, as for instance at the Large Hadron Collider (LHC) at CERN, the amount and the complexity of the data will significantly exceed those of today's experiments. This will put very strong demands on the speed and the robustness of the track finding and track fitting algorithms which are to be applied at LHC.

Recently, several novel algorithms have been developed, specifically designed to meet the requirements of the experiments of the LHC era. Some of these focus on robustness with respect to noise [1], while others have been shown to be very fast but nevertheless virtually as precise as optimal, time-consuming approaches for all but the lowest momenta. Estimation of the parameters of circular tracks by the Riemann sphere method [2] belongs to the latter category.

A major shortcoming of the Riemann fit method until now has been the lack of a method of deriving an expression of the covariance matrix of the estimated track parameters. We present in this paper such a method. The calculation of the covariance matrix turns out to be straightforward given the knowledge of the eigenvalues and the eigenvectors of a weighted sample covariance matrix of the measurements mapped onto the Riemann sphere, and these quantities are already available after the fit has been performed.

The paper is organized as follows. In the next section, we give a short review of the Riemann sphere track fitting method. Thereafter, the covariance matrix of the estimated parameters of the Riemann fit is derived. In Section 4, we state results of a simulation experiment from the ATLAS Transition Radiation Tracker, and the paper is concluded by a discussion and a short outlook to further research.

*Corresponding author. Fax: +47-611-35240.

E-mail addresses: are.strandlie@hig.no (A. Strandlie), fruhwirth@hephy.oeaw.ac.at (R. Frühwirth).

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2. Track fitting by the Riemann sphere method

This method of fitting circular particle tracks has been developed recently [2]. It is based on the idea of mapping the measurements onto the *Riemann sphere* [3] and fitting a plane to the transformed measurements. In this way, the nonlinear problem of circle fitting is transformed into a linear one. The mapping of the measurement (R_i, ϕ_i) (with $i = 1, \ldots, N$) onto the Riemann sphere is defined by

$$x_i = R_i \cos \phi_i / (1 + R_i^2) \tag{1}$$

$$y_i = R_i \sin \phi_i / (1 + R_i^2) \tag{2}$$

$$z_i = R_i^2 / (1 + R_i^2) \tag{3}$$

and the fitted plane is defined as the plane minimizing the cost function

$$S = \sum_{i=1}^{N} (1 + R_i^2)^2 d_i^2 = \sum_{i=1}^{N} p_i d_i^2 \tag{4}$$

where d_i is the distance from measurement i to the plane. In other words, we want to find the minimum of S with respect to the parameters $\boldsymbol{n}$ (normal vector of the plane) and c (signed distance from the plane to the origin). It turns out that the fitted normal vector is given as one of the eigenvectors of a weighted sample covariance matrix $\boldsymbol{A}_\mathrm{w}$ of the measurements,

$$\boldsymbol{A}_\mathrm{w} = \frac{1}{N} \sum_{i=1}^{N} p_i (\boldsymbol{r}_i - \boldsymbol{r}_\mathrm{cg})(\boldsymbol{r}_i - \boldsymbol{r}_\mathrm{cg})^\mathrm{T} \tag{5}$$

where $\boldsymbol{r}_i^\mathrm{T} = (x_i, y_i, z_i)$ and $\boldsymbol{r}_\mathrm{cg}^\mathrm{T} = (x_\mathrm{cg}, y_\mathrm{cg}, z_\mathrm{cg})$, with

$$x_\mathrm{cg} = \frac{\sum_{i=1}^{N} p_i x_i}{\sum_{j=1}^{N} p_j} \tag{6}$$

$$y_\mathrm{cg} = \frac{\sum_{i=1}^{N} p_i y_i}{\sum_{j=1}^{N} p_j} \tag{7}$$

$$z_\mathrm{cg} = \frac{\sum_{i=1}^{N} p_i z_i}{\sum_{j=1}^{N} p_j}. \tag{8}$$

The specific eigenvector to be chosen is the eigenvector corresponding to the smallest eigenvalue of $\boldsymbol{A}_\mathrm{w}$. With $\boldsymbol{n}$ given, c is straightforwardly found by

$$c = -\boldsymbol{n}^\mathrm{T} \boldsymbol{r}_\mathrm{cg}. \tag{9}$$

The mapping from the parameters $\boldsymbol{n}^\mathrm{T} = (n_1, n_2, n_3)$ and c down to the centre coordinates (u_0, v_0) and radius of curvature ρ of the circle is given by

$$u_0 = -\frac{n_1}{2(c + n_3)} \tag{10}$$

$$v_0 = -\frac{n_2}{2(c + n_3)} \tag{11}$$

$$\rho^2 = \frac{n_1^2 + n_2^2 - 4c(c + n_3)}{4(c + n_3)^2}. \tag{12}$$

Note that there is a singularity for $c = -n_3$, something which happens in the straight line limit. However, in the case where c is very close or equal to $-n_3$ it is possible to map the parameters of the plane to another set of circle parameters, say, the curvature κ, the angle ψ between the tangent to the track and the x-axis, and the distance from the origin a_0 at the point of closest approach to the origin. The latter set of parameters is also well defined in the limit of straight lines.

3. Derivation of the covariance matrix

In order to find an expression of the covariance matrix of the circle parameters, we first need to know the covariance matrix of the normal vector. To our knowledge, no exact expressions of this kind exist in the literature. However, in the asymptotic case, i.e. when the number of measurements becomes very large, an expression is available in the book by Mardia et al. [4]. According to them, the normalized eigenvector i of the weighted sample covariance matrix $\boldsymbol{A}_\mathrm{w}$ has asymptotically the covariance matrix $\boldsymbol{V}_i$, where

$$\boldsymbol{V}_i = \frac{\lambda_i}{N} \sum_{j \neq i} \frac{\lambda_j}{(\lambda_j - \lambda_i)^2} \gamma_j \gamma_j^\mathrm{T}. \tag{13}$$

Here N is the number of measurements and λ_i and γ_i are the ith eigenvalue and eigenvector of $\boldsymbol{A}_\mathrm{w}$, respectively. In order to determine the covariance matrix $\boldsymbol{C}_{\boldsymbol{n}}$ of the normal vector, i must be chosen as the label attached to the smallest eigenvalue.

Due to the normalization constraint of the normal vector, $\boldsymbol{C}_{\boldsymbol{n}}$ has rank two.

The basic assumption in this paper is that the above expression is approximately valid also for a finite number of measurements, and the simulation experiments of Section 4 will show that this is indeed the case. However, in the finite sample case it is not obvious that the normalization factor N of the covariance matrix should be exactly equal to the number of measurements. Our approach is to treat this normalization factor as a free parameter and use the simulations as a means of optimizing its value.

We will now proceed to the problem of calculating the full, four-dimensional covariance matrix of the parameters of the fitted plane, $\boldsymbol{C}_{(c,\boldsymbol{n})}$. Knowing $\boldsymbol{C}_{\boldsymbol{n}}$, the missing elements var(c) and cov($c, \boldsymbol{n}$) can be computed in a straightforward manner by using the lemma in the appendix. Note that $\boldsymbol{n}$ is a function of the sample covariance matrix $\boldsymbol{A}_{\mathrm{w}}$ which in the Gaussian model is independent of the sample mean $\boldsymbol{r}_{\mathrm{cg}}$. In the Gaussian model the assumptions of the lemma are therefore satisfied.

From the definition of c (Eq. (9)) it follows that

$$\mathrm{var}(c) = \mathrm{E}(\boldsymbol{r}_{\mathrm{cg}})^{\mathrm{T}} \boldsymbol{C}_{\boldsymbol{n}} \mathrm{E}(\boldsymbol{r}_{\mathrm{cg}}) + \mathrm{E}(\boldsymbol{n})^{\mathrm{T}} \boldsymbol{C}_{\boldsymbol{r}} \mathrm{E}(\boldsymbol{n}) + \mathrm{trace}(\boldsymbol{C}_{\boldsymbol{n}} \boldsymbol{C}_{\boldsymbol{r}}) \tag{14}$$

$$\mathrm{cov}(c, \boldsymbol{n}) = -\boldsymbol{C}_{\boldsymbol{n}} \mathrm{E}(\boldsymbol{r}_{\mathrm{cg}}). \tag{15}$$

As the expectations are unknown they are approximated by the observed vectors $\boldsymbol{n}$ and $\boldsymbol{r}_{\mathrm{cg}}$. It remains to compute the covariance matrix $\boldsymbol{C}_{\boldsymbol{r}} = \mathrm{var}(\boldsymbol{r}_{\mathrm{cg}})$. This is done by linear error propagation under the assumption that the R_i are known to infinite precision. It is therefore sufficient to compute the covariance matrix of $(x_{\mathrm{cg}}, y_{\mathrm{cg}})^{\mathrm{T}}$. The result is

$$\mathrm{var}[(x_{\mathrm{cg}}, y_{\mathrm{cg}})^{\mathrm{T}}] = \frac{1}{(\sum_{j=1}^{N} p_j)^2} \sum_{i=1}^{N} p_i^2 \mathrm{var}(\phi_i) \times \begin{pmatrix} y_i^2 & -x_i y_i \\ -x_i y_i & x_i^2 \end{pmatrix}. \tag{16}$$

This 2×2 matrix is padded with zeros in order to obtain the matrix $\boldsymbol{C}_{\boldsymbol{r}}$. The full covariance matrix $\boldsymbol{C}_{(c,\boldsymbol{n})}$ has dimension four and rank three.

The final step in our derivations is to calculate the covariance matrix of the circle parameters from $\boldsymbol{C}_{(c,\boldsymbol{n})}$, and this again can be done by linear error propagation. If we choose ψ, κ and a_0 as the set of parameters of the circle, the relation is

$$\boldsymbol{C}_{(\psi,\kappa,a_0)} = \boldsymbol{J} \cdot \boldsymbol{C}_{(c,\boldsymbol{n})} \cdot \boldsymbol{J}^{\mathrm{T}} \tag{17}$$

where $\boldsymbol{J}$ is the Jacobian of the transformation,

$$\boldsymbol{J} = \frac{\partial(\psi, \kappa, a_0)}{\partial(c, \boldsymbol{n})}. \tag{18}$$

Since $\boldsymbol{C}_{(c,\boldsymbol{n})}$ has rank 3, $\boldsymbol{C}_{(\psi,\kappa,a_0)}$ also has rank 3. In order to have a consistent set of parameters for all tracks, the curvature κ and the impact parameter a_0 have to be signed quantities. We define κ to be positive when the track has a clockwise rotation with respect to the positive z-axis, and a_0 to be positive when the z component of the angular momentum of the reconstructed track is negative. In general, the correct signs have to be determined from the knowledge of the direction of the track. Using these definitions, the mapping from the Riemann sphere parameters can be stated in the following way:

$$\psi = \arctan\left(\frac{n_2}{n_1}\right) \tag{19}$$

$$\kappa = s \frac{2(c + n_3)}{\sqrt{1 - n_3^2 - 4c(c + n_3)}} \tag{20}$$

$$a_0 = s \frac{\sqrt{1 - n_3^2 - 4c(c + n_3)} - \sqrt{1 - n_3^2}}{2(c + n_3)}. \tag{21}$$

Here s is equal to the product of the sign of κ and the sign of $c + n_3$. It can also be noted that we apply a four-quadrant version of the inverse tangent in the calculation of ψ, something which also requires the knowledge of the actual direction of the track. However, the magnitudes and signs of the derivatives of ψ are the same as for the standard, two-quadrant version of the inverse tangent. By calculating the derivatives, the non-zero elements of the Jacobian are given as

$$\frac{\partial \psi}{\partial n_1} = -\frac{n_2}{n_1^2 + n_2^2} \tag{22}$$

$$\frac{\partial \psi}{\partial n_2} = \frac{n_1}{n_1^2 + n_2^2} \tag{23}$$

$$\frac{\partial \kappa}{\partial n_3} = s\left(\frac{2}{\sqrt{1 - n_3^2 - 4c(c+n_3)}} + \frac{2(c+n_3)(2c+n_3)}{(1 - n_3^2 - 4c(c+n_3))^{3/2}}\right) \tag{24}$$

$$\frac{\partial \kappa}{\partial c} = s\left(\frac{2}{\sqrt{1 - n_3^2 - 4c(c+n_3)}} + \frac{4(c+n_3)(2c+n_3)}{(1 - n_3^2 - 4c(c+n_3))^{3/2}}\right) \tag{25}$$

$$\frac{\partial a_0}{\partial n_3} = s\left(\frac{\sqrt{1-n_3^2} - \sqrt{1 - n_3^2 - 4c(c+n_3)}}{2(c+n_3)^2} + \frac{n_3}{2(c+n_3)\sqrt{1-n_3^2}} - \frac{2c+n_3}{2(c+n_3)\sqrt{1 - n_3^2 - 4c(c+n_3)}}\right) \tag{26}$$

$$\frac{\partial a_0}{\partial c} = s\left(\frac{\sqrt{1-n_3^2} - \sqrt{1 - n_3^2 - 4c(c+n_3)}}{2(c+n_3)^2} - \frac{2c+n_3}{(c+n_3)\sqrt{1 - n_3^2 - 4c(c+n_3)}}\right). \tag{27}$$

In the case where a track is almost straight, there are potential singularities in the above expressions of the impact parameter and its derivatives. However, by series expansions of the square roots up to second order, the formulas become in this limit:

$$a_0 = -s(c(1 + \tfrac{1}{2}n_3^2) + c^2(c + n_3)) \tag{28}$$

$$\frac{\partial a_0}{\partial n_3} = -sc(c + n_3) \tag{29}$$

$$\frac{\partial a_0}{\partial c} = -s\left(1 + \frac{1}{2}n_3^2 + 3c^2 + 2n_3 c\right). \tag{30}$$

These expressions are well defined also in the limit $c + n_3 \to 0$.

4. A simulation experiment in the ATLAS TRT

The ATLAS Transition Radiation Tracker (TRT) is the outermost part of the Inner Detector of the ATLAS experiment at the LHC at CERN, and we have focused on simulated tracks coming from the barrel part of the TRT. The TRT is a drift tube detector with in-principle ambiguous measurements. In our simulations, however, we have turned the mirror hits off. Also, since the focus of this paper is track fitting rather than track finding, we assume that a 100% efficient pattern recognition procedure has been applied beforehand, i.e. there are no noise hits in the track candidates. There are 9800 tracks in our track sample, all with a transversal momentum above 1 GeV/c. The track sample used in this work is the same as the one described in Ref. [2]. For more details of the TRT, the reader is referred to the ATLAS Inner Detector TDR [5].

We have first tested out how well $\boldsymbol{C}_{(c,\boldsymbol{n})}$ reflects the actual spread of the parameters of the fitted plane. This has been done by looking at the standardized residuals, i.e. the residuals of the estimated parameters with respect to the true values divided by the estimated standard deviations, and the tail probabilities. The latter quantity is defined as one minus the cumulative distribution function of the following:

$$\chi^2 = \Delta\boldsymbol{p}^{\mathrm{T}} \cdot \boldsymbol{C}_{(c,\boldsymbol{n})}^{-1} \cdot \Delta\boldsymbol{p} \tag{31}$$

where $\Delta\boldsymbol{p}$ is the vector consisting of the residuals of the estimated parameters with respect to the true ones. If this χ^2 really obeys a χ^2-distribution, a histogram consisting of all the tail probabilities should be reasonably flat. Since $\boldsymbol{C}_{(c,\boldsymbol{n})}$ does not have full rank, the inverse does not exist. We have therefore used a generalized inverse in the calculations of this χ^2.

The relevant histograms are shown in Fig. 1. As can be seen in (a), the histogram of the tail

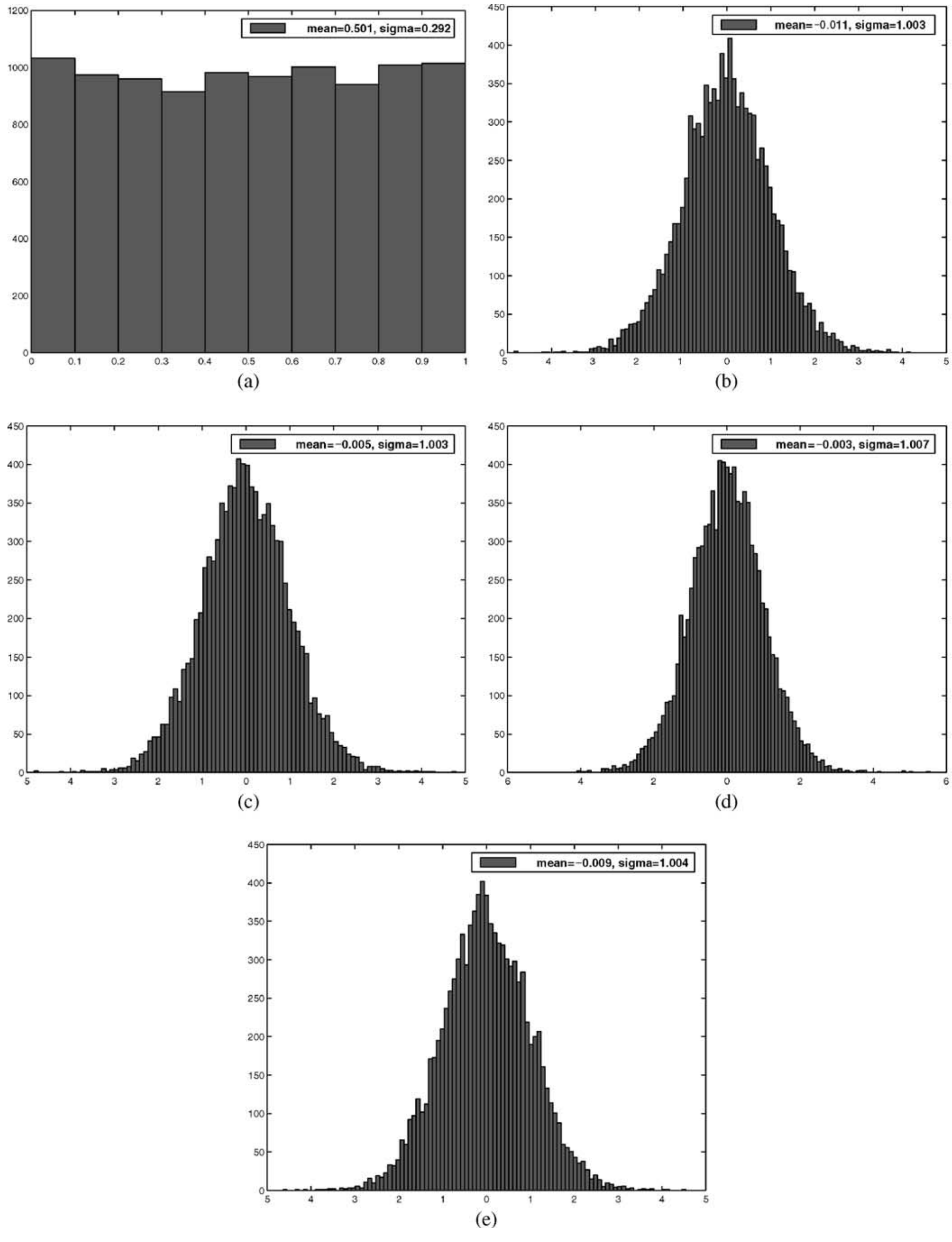

Fig. 1. Histograms of tail probabilities (a) and standardized residuals of c, n_1, n_2 and n_3 (b)–(e).

probabilities is reasonably flat, and its mean value is 0.501. The standardized residuals (shown in (b)–(e)) have standard deviations ranging from 1.003 to 1.007, i.e. they are all very close to unity. These tests establish the fact that the estimated covariance matrix very well reproduces the actual spread of the parameters. It can be noted that the optimal value of the normalization factor, defined in Eq. (13), turns out to be $N-5$, where N is the number of measurements

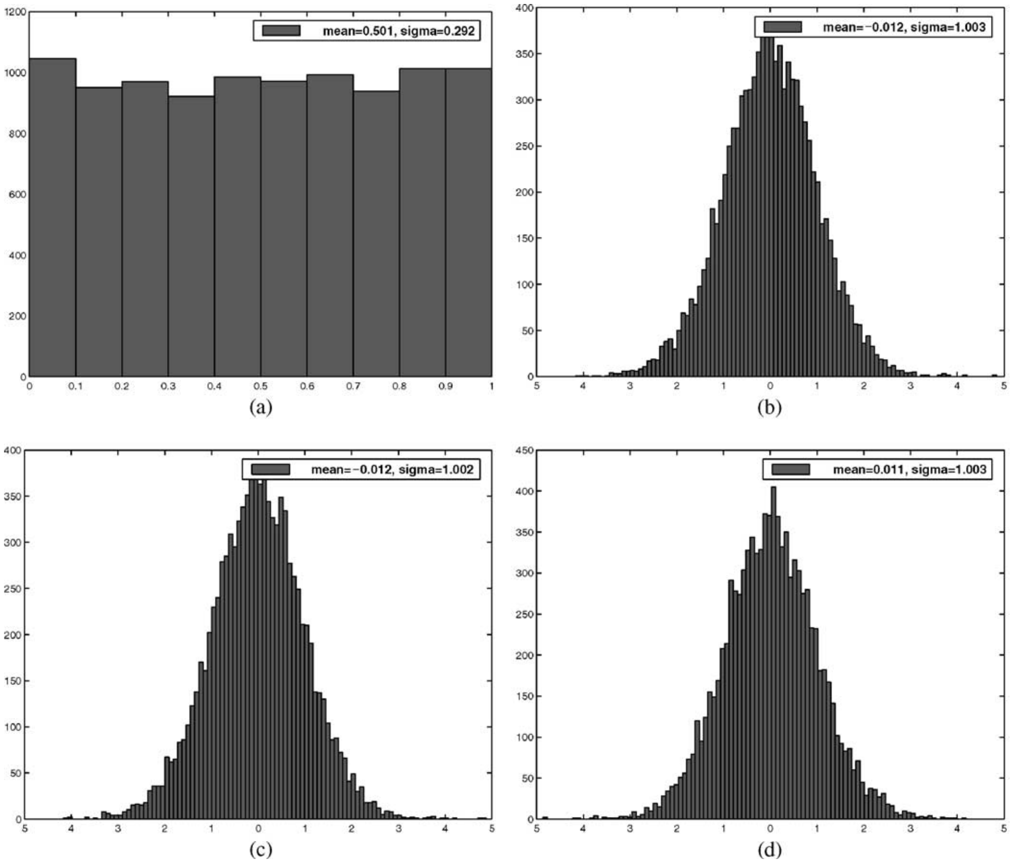

Fig. 2. Histograms of tail probabilities (a) and standardized residuals of ψ, κ and a_0 (b)–(d).

in the track candidate. In the ATLAS TRT, N is about 35.

In Fig. 2, we show quantities similar to the ones in Fig. 1 for the circle parameters ψ, κ and a_0. The mean value of the tail probabilities in (a) is still 0.501, and the standard deviations of the standardized residuals (shown in (b)–(d)) are all about 1.003. The quality of the linear approximation in the propagation of the errors is thus obviously more than good enough.

5. Discussion and outlook

We have in this paper presented a method of deriving a covariance matrix of the estimated parameters of circular tracks fitted by the Riemann sphere method. The calculation of the covariance matrix is straightforward, given the knowledge of the eigenvalues and eigenvectors of a weighted sample covariance matrix of the measurements mapped onto the Riemann sphere. Through a simulation experiment from the ATLAS TRT, the covariance matrix presented in this paper has been shown to very well reproduce the actual spread of the track parameters.

It can be noted that the Riemann fit method is quite similar to the Karimäki method of fitting circular arcs [6]. This is due to the fact that even though the two methods are derived in different ways, they both amount to a least-squares fit of the track parameters. We have tested out the Karimäki method on the simulated data presented in Section 4. A comparison with the Riemann fit shows that the performance of the two methods is virtually identical, concerning both the accuracy of the estimates and the computational speed. The Riemann fit is, however, more general than the Karimäki method because it is not necessary to assume that the observation error is small compared to the radius of the circle.

The Riemann fit method is restricted to two-dimensional data, i.e. it works only in the case where the data are truly two-dimensional, as in the ATLAS TRT, or in the case where three-dimensional data are projected into the bending plane. The need for generalizing the method to truly three-dimensional data is therefore obvious. Progress in this direction is being made, and we intend to report results from this ongoing work in a subsequent paper.

Appendix

Lemma. *Assume that* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are n-dimensional independent stochastic vectors with expectations* $\boldsymbol{\alpha} = \mathrm{E}(\boldsymbol{a})$ *and* $\boldsymbol{\beta} = \mathrm{E}(\boldsymbol{b})$ *and covariance matrices* $\boldsymbol{A} = \mathrm{var}(\boldsymbol{a})$ *and* $\boldsymbol{B} = \mathrm{var}(\boldsymbol{b})$, *respectively. Let* $c = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}$. *Then*

(a) $\mathrm{E}(c) = \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\beta}$
(b) $\mathrm{var}(c) = \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{\alpha} + \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{\beta} + \mathrm{trace}(\boldsymbol{AB})$
(c) $\mathrm{cov}(\boldsymbol{a}, c) = \boldsymbol{A}\boldsymbol{\beta}$.

Proof. (a) Follows immediately from the independency of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\begin{aligned}
\text{(b)}\ \ \mathrm{var}(c) &= \mathrm{E}\left[\left(\textstyle\sum_i a_i b_i\right)^2\right] - \left[\textstyle\sum_i \mathrm{E}(a_i)\mathrm{E}(b_i)\right]^2 \\
&= \textstyle\sum_i \sum_j [\mathrm{E}(a_i a_j)\mathrm{E}(b_i b_j) - \alpha_i\alpha_j\beta_i\beta_j] \\
&= \textstyle\sum_i \sum_j [(A_{ij} + \alpha_i\alpha_j)(B_{ij} + \beta_i\beta_j) \\
&\quad - \alpha_i\alpha_j\beta_i\beta_j] \\
&= \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{\alpha} + \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{\beta} + \mathrm{trace}(\boldsymbol{AB}).
\end{aligned}$$

$$\begin{aligned}
\text{(c)}\ \ \mathrm{cov}(\boldsymbol{a}, c) &= \mathrm{E}(\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}) - \mathrm{E}(\boldsymbol{a})\mathrm{E}(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}) \\
&= [\mathrm{E}(\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}) - \mathrm{E}(\boldsymbol{a})\mathrm{E}(\boldsymbol{a}^{\mathrm{T}})]\mathrm{E}(\boldsymbol{b}) = \boldsymbol{A}\boldsymbol{\beta}.
\end{aligned}$$

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